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Kalman filtering with synthetic measurements under an event-triggered sensor scheduler

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Abstract— **This work studies the information update of the Kalman filter under a threshold-based event-triggered sensor scheduler designed to reduce the sensor-to-estimator communication cost while preserving estimation accuracy. For each sensor, when its normalized innovation is below a threshold required for data transmission (i.e., the sensor does not send measurements to the estimator), existing filtering algorithms extract this implicit information to update the estimation error covariance. However, when the low normalized innovations are insufficient indicators of the overall estimation accuracy, the state estimate may still need to be corrected. This occurs for example when the sensors directly measure only a subset of the full state vector.**

We propose a filtering algorithm to correct the state estimate in addition to the error covariance without requiring additional data transmission. The estimator performs the correction using synthetic measurements with bounded error compared to the true measurements, which are generated by the estimator. By correcting the estimate with synthetic measurements, the proposed filter can further reduce the estimation error at a small cost of error covariance inflation. We first show that the proposed filter is an approximate minimum mean square error estimator when the synthetic measurement is given. Second, the estimation error dynamics of the proposed filter is shown to be input-to-state stable, indicating that the estimation error is also small if the disparity between the synthetic and true measurements is small. Numerical experiments illustrate the evolution of the estimation error given by the proposed filter, and show the filter can improve the overall estimation accuracy. Supplementary source code is available at https://github.com/yesun/KFSMecc2016.

I. INTRODUCTION

The pervasiveness of wireless sensor networks has revolutionized real-time monitoring systems. However, in many *remote estimation* problems, the cost of data transmission between sensors and the estimator (e.g., energy and channel bandwidth costs) becomes a design concern. An approach to solve the remote estimation problem has been through *sensor scheduling* techniques, which determine when the sensor data is most informative for state estimation and transmit it accordingly (e.g., see [1] and references therein). Such schemes enable satisfaction of estimation accuracy while meeting the communication budget constraints.

In a centralized sensor scheduling scheme [2]–[6], the optimal sensor selection strategies are developed from the estimator perspective, assuming that the estimator can obtain data at any time from any sensor it queries. Comparatively, in a decentralized sensor scheduling scheme, data transmission decisions are made locally at the sensors. For example, in *delta sampling* (e.g., [7]), a new measurement is transmitted when it moves away from the previously transmitted data by a distance delta. *Relevant sampling* is proposed in [8] which triggers an event to send relevant measurements to the estimator (i.e., measurements that contribute to reducing the estimator uncertainty and the estimation error). Data transmission decisions can also be computed by minimizing a cost function consisting of the expected estimation error and the data transmission cost (e.g., [9], [10]), or by integrating physical constraints on the sensors [11], [12]. In [9]–[11], [13]–[15], the event that triggers transmission of sensor measurements is the fact that the gap between the true and predicted measurements exceeds a threshold, which is very close to the sensor scheduling criteria applied in this work.

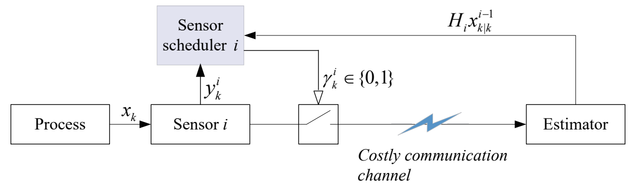

Fig. 1. Remote estimation with an event-triggered sensor scheduler, shown for a single sensor i.

This article considers a remote estimation problem composed of a central estimator and multiple remote sensors, with the communication topology for a single sensor illustrated in Figure 1. It is assumed that for the sensors, transmitting data is costly, while receiving data is relatively cheap. This is partially motivated by the fact that the power amplifier applied when sending data is the dominant source of energy consumption in many wireless communication technologies [16]. The resources on the central estimator's side are considered to be sufficient, while energy at the sensor nodes is limited. Hence, the budget on sensor-to-estimator communications needs to be smartly allocated.

The target to be tracked evolves as the following linear time-varying system

$$x_{k+1} = A_k x_k + w_k, \quad x_k \in \mathbb{R}^n, \tag{1}$$

where w_k is the white Gaussian model noise following distribution $w_k \sim \mathcal{N}(\mathbf{0}, Q_k)$, with a positive definite error covariance matrix $Q_k > 0$. It is assumed that the total number of possible matrices A_k and Q_k are finite. Let S be the total number of sensors in the system. For $i \in \mathcal{S} = \{1, \cdots, S\}$, the sensor measurement y^i_k from sensor i at time

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k is modeled by the following linear observation equation

$$y_k^i = H^i x_k + v_k^i, \quad y_k^i \in \mathbb{R}^{m^i}, \tag{2}$$

where H^i is the time-invariant[1] observation matrix of sensor i, and $v_k^i \sim \mathcal{N}(\mathbf{0}, R^i)$ is the white Gaussian measurement noise with covariance $R^i > 0$, and is independent of the model noise. Construct $H = \left(H^{1\top}, \cdots, H^{S\top}\right)^\top$ as the observation matrix of the entire sensor set. For all $k \geq 0$, the matrix pairs (A_k, Q_k) and (A_k, H) are assumed to be controllable and observable, respectively. For $i \in \mathcal{S}$, denote as $x_{k|k}^i$ the state estimate of x_k after the information from sensor i at time k (either the sensor measurement or the fact that data transmission did not occur) is processed by the estimator, and $\Gamma_{k|k}^i$ the error covariance associated with $x_{k|k}^i$. Denote as $x_{k|k-1}$ the estimate of x_k before any information obtained at time k from the sensors is processed, and $\Gamma_{k|k-1}$ the error covariance of $x_{k|k-1}$. At each time step k, the scheduler at sensor i makes a transmission decision based on the normalized disparity between $H^i x_{k|k}^{i-1}$ (the predicted measurement of sensor i given $x_{k|k}^{i-1}$) broadcast from the estimator and y_k^i measured at sensor i. If the normalized disparity exceeds a given threshold, sensor data y_k^i is transmitted to the central estimator. The decision variable $\gamma_k^i \in \{0, 1\}$ indicates if y_k^i is sent ($\gamma_k^i = 1$) or not sent ($\gamma_k^i = 0$).

Sensor schedulers provide additional implicit information to the central estimator when no data is sent. The state-of-the-art filtering algorithms [17]–[21] developed to solve the remote estimation problem illustrated in Figure 1 leverage this implicit information and update the estimate recursively according to the following framework:

Time update:

$$\left\{\begin{array}{l}\text{Update the state estimate } x_{k-1|k-1}^S \to x_{k|k-1} \\ \text{Update the error covariance } \Gamma_{k-1|k-1}^S \to \Gamma_{k|k-1},\end{array}\right. \tag{3}$$

Information update:

$$\left\{\begin{array}{l}\text{Let } x_{k|k}^0 = x_{k|k-1} \text{ and } \Gamma_{k|k}^0 = \Gamma_{k|k-1} \\ \text{For } i = 1 \text{ to } S \text{ do} \\ \quad \text{Sensor scheduler } i \text{ computes sensor decision } \gamma_k^i \\ \quad \text{If } \gamma_k^i = 1 \\ \quad\quad \text{Update } x_{k|k}^{i-1} \to x_{k|k}^i \text{ and } \Gamma_{k|k}^{i-1} \to \Gamma_{k|k}^i \\ \quad\quad\; \text{by the standard Kalman filter} \\ \quad \text{Else} \\ \quad\quad \text{Set } x_{k|k}^i = x_{k|k}^{i-1} \\ \quad\quad \text{Update the error covariance } \Gamma_{k|k}^{i-1} \to \Gamma_{k|k}^i.\end{array}\right. \tag{4}$$

In summary, when a sensor does not send data, only the error covariance is updated when the estimator processes the information from this sensor, and the precise update formula depends on the sensor scheduler. The state estimates, however, are not updated.

A. Motivation

Although framework (3)-(4) is proposed to minimize the volume of the non-transmission region of the sensor measurements [17] or the mean square error [18]–[21], the information update in (4) shows the state estimate is not corrected when data transmission is declined. As a consequence, the error of the state estimate cannot be reduced (although the estimation error covariance is reduced), unless the true state converges asymptotically to zero. However, in many applications, the sensors altogether measure only a subset of the state variables, thus the sensor schedulers may not serve as a sufficient indicator of the estimation accuracy. For example, consider a physical system tracked by a set of sensors which are distributed sparsely compared to the dimension of the state. Even if the disparity between the sensor measurement and the predicted estimate is below the threshold required for data transmission, the estimate of the full state vector may be inaccurate and benefit from a measurement correction in the information update.

Given the above concerns, this article proposes a filter which always corrects the state estimate, even without data transmission. When y_k^i is not sent, the state estimate is corrected via a synthetic measurement $\tilde{y}_k^i$ (generated based on the estimated distribution of the true measurement) as follows:

$$x_{k|k}^i = x_{k|k}^{i-1} + \tilde{K}_k^i \left(\tilde{y}_k^i - H^i x_{k|k}^{i-1}\right), \tag{5}$$

where $\tilde{K}_k^i$ is the synthetic gain associated with $\tilde{y}_k^i$. When data transmission is triggered, the state estimate is corrected by the true measurement in the information update. Consequently, the update framework studied in this work reads:

Time update: See (3)

Information update:

$$\left\{\begin{array}{l}\text{Let } x_{k|k}^0 = x_{k|k-1} \text{ and } \Gamma_{k|k}^0 = \Gamma_{k|k-1} \\ \text{For } i = 1 \text{ to } S \text{ do} \\ \quad \text{Sensor scheduler } i \text{ computes sensor decision } \gamma_k^i \\ \quad \text{If } \gamma_k^i = 1 \\ \quad\quad \text{Update } x_{k|k}^{i-1} \to x_{k|k}^i \text{ and } \Gamma_{k|k}^{i-1} \to \Gamma_{k|k}^i \\ \quad\quad\; \text{by the standard Kalman filter} \\ \quad \text{Else} \\ \quad\quad \text{Generate synthetic measurement } \tilde{y}_k^i \\ \quad\quad \text{Update } x_{k|k}^{i-1} \to x_{k|k}^i \text{ via } \tilde{y}_k^i \text{ based on (5)} \\ \quad\quad \text{Update the error covariance } \Gamma_{k|k}^{i-1} \to \Gamma_{k|k}^i,\end{array}\right. \tag{6}$$

Under information update (5), the estimation error can continue to decrease even when no data is sent. Even if the synthetic measurement is not close enough to the true measurement, the presence of measurement feedback itself outperforms no feedback at all. This is due to the fact that the disparity between the synthetic and the true measurements is designed to be within a bound set by the sensor scheduler. To

[1]The time-invariance of the observation equation for each sensor is motivated by the practical concern that information transmission from the sensors to the estimator is expensive. When the observation equation is time-varying, even if sensor data y_k^i is not sent, sensor i still needs to send costly information on H_k^i and R_k^i to the estimator at each time step k (so that the estimator can update the estimation error covariance), which conflicts with the original goal of sensor scheduling to reduce communication costs.

justify the effectiveness of the synthetic measurements, we compare numerically the performance of the proposed filter with various filters designed under framework (3)-(4), under the same sensor-to-estimator communication rate defined by:

$$r = \frac{1}{K|\mathcal{S}|} \sum_{k=1}^{K} \sum_{i \in \mathcal{S}} \gamma_k^i, \tag{7}$$

where K is the total number of time steps, and $|\mathcal{S}|$ is the total number of sensors.

To the best of our knowledge, this framework has not been studied in the existing literature. Although [17] also propose to compute $x_{k|k}^i$ according to (5) based on a virtual measurement as a supplement to the true measurement when $\gamma_k^i = 0$, the proposed virtual measurement is deterministic and is set as $\tilde{y}_k^i = H^i x_{k|k}^{i-1}$. This implies that there is no difference between the virtual measurement and the predicted measurement given the latest estimate $x_{k|k}^{i-1}$, thus the resulting $x_{k|k}^i$ computed from (5) is exactly the same as $x_{k|k}^{i-1}$. Consequently, the proposed estimator in [17] fits framework (3)-(4).

B. Contributions

In this work, a *Kalman filter with synthetic measurements* (KF-SM) is designed and analysed which updates the estimate according to (3) and (6), with the scheduling scheme chosen to be a deterministic threshold-based sensor scheduler (the threshold applied in the sensor scheduler is predefined and known by the estimator). Explicitly, when sensor i does not send data y_k^i (i.e., $\gamma_k^i = 0$), we propose a synthetic measurement generation algorithm that extracts the implicit information provided by the sensor scheduler and outputs a synthetic measurement $\tilde{y}_k^i$. Hence, the state estimate is corrected according to (5) (Section III). The synthetic measurement $\tilde{y}_k^i$ and the gain $\tilde{K}_k^i$ are chosen such that the estimation error dynamics as well as the estimation error covariance of the KF-SM are stable. When the true measurement y_k^i is sent, the information update is the same as the standard *Kalman filter* (KF). The properties of the KF-SM introduced in this article include:

1) The KF-SM is an approximate MMSE estimator when the synthetic measurement is given (Section IV-A).
2) The estimation error dynamics of the KF-SM is input-to-state stable when treating the synthetic measurement noise (with respect to the true measurement) as an input to the estimation error dynamics, thus the error dynamics is ultimately bounded (Section IV-B).

The advantages of the KF-SM in reducing the estimation error are verified numerically in Section V.

II. PRELIMINARIES

A. The Sequential Processing Form of the Kalman Filter

Recall from Section I-A that $\gamma_k^i \in \{0, 1\}$ is the decision variable indicating if sensor data y_k^i will be transmitted to the estimator. Denote as $\gamma_k = \left(\gamma_k^1, \cdots, \gamma_k^S\right)^\top$ the decision variables at time k, and y_k the vector with elements y_k^i for $i \in \mathcal{J}_k$, where the set $\mathcal{J}_k = \{i | \gamma_k^i = 1, i \in \mathcal{S}\}$ defines the set of sensors that transmit data at time k. The information set containing the decision variables and sensor measurements up to time $k-1$ is denoted by $\mathcal{I}_{k-1} = \{\gamma_0, \cdots, \gamma_{k-1}, y_0, \cdots, y_{k-1}\}$, which is available only to the central estimator. Furthermore, define the information set obtained by the estimator at time k based on data from the first i sensors and all the past data as

$$\mathcal{I}_k^i = \begin{cases} \mathcal{I}_k^{i-1} \cup \{\gamma_k^i\} & \text{if } \gamma_k^i = 0, \\ \mathcal{I}_k^{i-1} \cup \{\gamma_k^i, y_k^i\} & \text{if } \gamma_k^i = 1, \end{cases}$$

for $i \in \mathcal{S}$ with $\mathcal{I}_k^0 = \mathcal{I}_{k-1}$, where $\mathcal{I}_{-1} = \emptyset$.

The central estimator computes the state estimate given the past information transmitted from the sensors. The *prior estimate* and *posterior estimate* of the state at time k can be expressed as $x_{k|k-1} = \mathbb{E}[x_k | \mathcal{I}_{k-1}]$ and $x_{k|k} = \mathbb{E}[x_k | \mathcal{I}_k]$, respectively. When the estimator conducts sequential processing of the sensor data (i.e., the measurements from multiple sensors are processed one sensor at a time within the same time step k), the intermediate estimates are defined as $x_{k|k}^i = \mathbb{E}[x_k | \mathcal{I}_k^i]$ for $i \in \mathcal{S}$, and the posterior estimate is then given by $x_{k|k} = x_{k|k}^S$. The estimation errors and the estimation error covariance matrices associated with the corresponding estimates defined above are given by

$$\begin{cases} \eta_{k|k-1} = x_{k|k-1} - x_k \\ \Gamma_{k|k-1} = \mathbb{E}\left[\left(x_{k|k-1} - x_k\right) \left(x_{k|k-1} - x_k\right)^\top \middle| \mathcal{I}_{k-1} \right], \end{cases}$$

$$\begin{cases} \eta_{k|k}^i = x_{k|k}^i - x_k \\ \Gamma_{k|k}^i = \mathbb{E}\left[\left(x_{k|k}^i - x_k\right) \left(x_{k|k}^i - x_k\right)^\top \middle| \mathcal{I}_k^i \right], \text{for } i \in \mathcal{S}, \end{cases}$$

$$\begin{cases} \eta_{k|k} = x_{k|k} - x_k \\ \Gamma_{k|k} = \mathbb{E}\left[\left(x_{k|k} - x_k\right) \left(x_{k|k} - x_k\right)^\top \middle| \mathcal{I}_k \right], \end{cases}$$

where the initial estimates on the state and the error covariance are given by $x_{0|-1}$ and $\Gamma_{0|-1}$, respectively.

Recall that when $\gamma_k^i = 1$ for all $i \in \mathcal{S}$ and $k \geq 0$, the standard KF is obtained, which recursively computes $x_{k|k}$ from $x_{k-1|k-1}$. The reader is referred to [22] for the following form of the KF, where the estimate is updated sequentially over the sensor data in the information update:

Time update:

$$\begin{cases} x_{k|k-1} = A_{k-1} x_{k-1|k-1} \\ \Gamma_{k|k-1} = A_{k-1} \Gamma_{k-1|k-1} A_{k-1}^\top + Q_{k-1}, \end{cases} \tag{8}$$

Information update:

$$\begin{cases} \text{Let } x_{k|k}^0 = x_{k|k-1}, \text{ and } \Gamma_{k|k}^0 = \Gamma_{k|k-1}, \\ \text{For } i = 1 \text{ to } S \text{ do} \\ \quad x_{k|k}^i = x_{k|k}^{i-1} + K_k^i \left(y_k^i - H^i x_{k|k}^{i-1} \right) \\ \quad \Gamma_{k|k}^i = \Gamma_{k|k}^{i-1} - K_k^i H^i \Gamma_{k|k}^{i-1} \\ \quad \text{where } K_k^i = \Gamma_{k|k}^{i-1} H^{i\top} \left(R^i + H^i \Gamma_{k|k}^{i-1} H^{i\top} \right)^{-1}, \\ x_{k|k} = x_{k|k}^S. \end{cases} \tag{9}$$

B. Kalman Filter with a Deterministic Threshold-based Sensor Scheduler

In the deterministic threshold-based sensor scheduler proposed in [18], [19], the sensor decision variable is computed as a function of the innovation $z_k^i = y_k^i - H^i x_{k|k}^{i-1}$ as follows:

$$\gamma_k^i = \begin{cases} 0 & \text{if } \phi(z_k^i) < \zeta^i \\ 1 & \text{if } \phi(z_k^i) \geq \zeta^i, \end{cases} \tag{10}$$

where ζ^i is a pre-determined deterministic threshold, and the function $\phi(z_k^i)$ is defined by the following equations:

$$\phi(z_k^i) = \left\| \left(G_k^i\right)^\top z_k^i \right\|_\infty = \|\epsilon_k^i\|_\infty, \tag{11}$$

where

$$G_k^i = \left(R^i + H^i \Gamma_{k|k}^{i-1} H^{i\top}\right)^{-\frac{1}{2}}, \tag{12}$$

and $\epsilon_k^i = \left(G_k^i\right)^\top z_k^i$ is defined to be the normalized innovation. The essence of a deterministic threshold-based sensor scheduler is that it assesses how much new information sensor data y_k^i can provide to the estimator by comparing the normalized innovation with a deterministic threshold, thus determining if y_k^i should be transmitted to the estimator.

Note that given the scheduling policy (10) where ζ^i is deterministic and known by the estimator, it holds that $\mathbb{P}(x_k = x, y_k^i = y | \mathcal{I}_k^{i-1}, \gamma_k^i = 0) = 0$ when $\phi(y - H^i x) \geq \zeta^i$. This indicates that $\mathbb{P}(x_k = x | \mathcal{I}_k^{i-1}, \gamma_k^i = 0)$ is not Gaussian since $y_k^i = H^i x_k + v_k^i$ as stated in (2). The exact MMSE estimator under the scheduling policy (10) is presented in [18], which is a nonlinear filter (since the sensor scheduler is a nonlinear function of the measurement), and is known to be computationally intractable [23]. Hence, the following Gaussian approximation[2]

$$\mathbb{P}(x_k = x | \mathcal{I}_k^{i-1}, \gamma_k^i = 0) \simeq \mathcal{N}(x_{k|k}^i, \Gamma_{k|k}^i), \quad \text{for } i \in \mathcal{S} \tag{13}$$

is widely used (see [18] and references therein) to reduce the problem of tracking the evolution of a general *probability density function* (PDF) to that of tracking its mean $x_{k|k}^i$ and covariance $\Gamma_{k|k}^i$, and to derive a recursive filtering algorithm. It is shown in [18] that under approximation (13), the normalized innovation follows $\epsilon_k^i \sim \mathcal{N}(\mathbf{0}, I_{m^i})$ before the sensor scheduler decides if y_k^i is transmitted to the estimator, where I_{m^i} is the identity matrix with dimension m^i. When the sensor scheduler decides not to send sensor data y_k^i, even though the measurement is not received, the estimator obtains additional information that $\|\epsilon_k^i\|_\infty < \zeta^i$, and the distribution of ϵ_k^i is updated from a normal distribution to a truncated normal distribution as follows:

$$\mathbb{P}\left(\epsilon_k^i = \epsilon \,\middle|\, \|\epsilon_k^i\|_\infty < \zeta^i\right) = \begin{cases} \frac{1}{p_{\zeta^i}} \mathbb{P}(\xi = \epsilon) & \text{if } \|\epsilon\|_\infty < \zeta^i \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where $\xi \sim \mathcal{N}(\mathbf{0}, I_{m^i})$ and $p_{\zeta^i} = \Pr\left(\|\xi\|_\infty < \zeta^i\right)$. In [18] and [19], the Kalman filter with a deterministic threshold-based sensor scheduler is derived based on the updated distribution (14), and is shown to be an approximate MMSE estimator, with the update equations given in the following definition.

Definition 1. The *Kalman filter with a deterministic threshold-based sensor scheduler* (KF-DT)[3] applies sensor scheduler (10)-(11), with update equations given by

Time update: See (8),

Information update:

$$\left\{\begin{array}{l} \text{Let } x_{k|k}^0 = x_{k|k-1}, \text{ and } \Gamma_{k|k}^0 = \Gamma_{k|k-1} \\ \text{For } i = 1 \text{ to } S \text{ do} \\ \quad \text{Sensor scheduler } i \text{ computes } \gamma_k^i \text{ based on (10)} \\ \quad \text{If } \gamma_k^i = 1 \\ \quad\quad \text{Compute } x_{k|k}^i \text{ and } \Gamma_{k|k}^i \text{ according to (9)} \\ \quad \text{Else} \\ \quad\quad x_{k|k}^i = x_{k|k}^{i-1} \\ \quad\quad \Gamma_{k|k}^i = \Gamma_{k|k}^{i-1} - K_k^i H^i \Gamma_{k|k}^{i-1} \\ \quad\quad \text{where } K_k^i = \chi\left(\zeta^i\right) \Gamma_{k|k}^{i-1} H^{i\top} (R^i + H^i \Gamma_{k|k}^{i-1} H^{i\top})^{-1}, \\ x_{k|k} = x_{k|k}^S, \end{array}\right. \tag{15}$$

with $0 < \chi(\zeta) < 1$ given by

$$\chi(\zeta) = \sqrt{\frac{2}{\pi}} \zeta \exp\left(-\frac{\zeta^2}{2}\right) \left(1 - 2Q(\zeta)\right)^{-1}, \tag{16}$$

where $Q(\zeta) = \int_\zeta^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$.

III. KALMAN FILTER WITH SYNTHETIC MEASUREMENTS

For the KF-DT, state estimate $x_{k|k}^i$ is not corrected in the information update when $\gamma_k^i = 0$. In this section, we propose a filtering algorithm that corrects both the state estimate and the estimation error covariance in the information update when data transmission does not occur. We apply sensor scheduler (10), which enables the estimator to draw synthetic measurements whose error with respect to the true measurements is guaranteed to be upper bounded by a function of the deterministic threshold ζ^i.

The same Gaussian assumption (13) is made for the KF-SM. Based on this assumption, the normalized innovation follows distribution (14) when $\gamma_k^i = 0$. The KF-SM aims to construct a feedback loop to update the state estimate by the linear information update (5), where the synthetic measurement $\tilde{y}_k^i$ is consistent with $\gamma_k^i = 0$ (i.e., the sensor scheduler would not trigger data transmission if the true measurement is $\tilde{y}_k^i$).

Given (5), one may note that if $\tilde{y}_k^i = H^i x_{k|k}^{i-1}$, the information update (5) is equivalent to letting $x_{k|k}^i = x_{k|k}^{i-1}$ when y_k^i is not sent, resulting in no correction on the state

[2]The error covariance $\Gamma_{k|k}^i$ in (13) is computed given the fact that $\gamma_k^i = 0$, with its explicit formula provided in (15).

[3]In [18], [19], the filter defined in Definition 1 is referred to as an approximate MMSE under the incorporated sensor scheduler, while the abbreviation "KF-DT" is introduced in this article to simplify the name.

estimate as described in framework (3)-(4). To reduce the probability that $\tilde{y}_k^i$ and $H^i x_{k|k}^{i-1}$ are close, the standard normal distribution $\epsilon_k^i \sim \mathcal{N}(\mathbf{0}, I_{m^i})$ is translated by a nonzero vector $\bar{\epsilon}_k^i$ which satisfies $\|\bar{\epsilon}_k^i\|_\infty < \zeta^i$, before imposing the condition $\|\tilde{\epsilon}_k^i\|_\infty < \zeta^i$ in the synthetic measurement generation algorithm presented in Algorithm 1. There are various ways to generate input $\bar{\epsilon}_k^i$, and one straightforward option is to draw $\bar{\epsilon}_k^i$ from distribution (14).

Algorithm 1 Synthetic Measurement Generation

Input: threshold ζ^i, matrix G_k^i defined in (12), vector $H^i x_{k|k}^{i-1}$ and $\bar{\epsilon}_k^i \neq \mathbf{0}$ such that $\|\bar{\epsilon}_k^i\|_\infty < \zeta^i$

Output: synthetic measurement $\tilde{y}_k^i$

1. Draw $\tilde{\epsilon}_k^i$ from the distribution $\mathbb{P}(\epsilon_k^i + \bar{\epsilon}_k^i = \epsilon)$ (i.e., the PDF of $\mathcal{N}(\mathbf{0}, I_{m^i})$ after translated by $\bar{\epsilon}_k^i$)
2. **while** $\|\tilde{\epsilon}_k^i\|_\infty \geq \zeta^i$ **do**
3. Redraw $\tilde{\epsilon}_k^i$ from $\mathbb{P}(\epsilon_k^i + \bar{\epsilon}_k^i = \epsilon)$
4. **end while**
5. Compute the synthetic innovation $\tilde{z}_k^i = \left(G_k^i\right)^{-1} \tilde{\epsilon}_k^i$
6. **return** synthetic measurement $\tilde{y}_k^i = \tilde{z}_k^i + H^i x_{k|k}^{i-1}$

One may note that based on Algorithm 1, it holds that $0 < \left\|\mathbb{E}\left[\tilde{\epsilon}_k^i\right]\right\|_\infty < \left\|\bar{\epsilon}_k^i\right\|_\infty < \zeta^i$. Denote as $u_k^i = \tilde{y}_k^i - y_k^i$ the disparity between the synthetic and true measurements. When $\|\tilde{\epsilon}_k^i\|_\infty < \zeta^i$, the disparity satisfies $\left\|u_k^i\right\| < 2\zeta^i \left\|G_k^i\right\|^{-1} (m^i)^{\frac{1}{2}}$. The KF-SM is defined below.

Definition 2. The *Kalman filter with synthetic measurements* applies sensor scheduler (10)-(11), with update equations given by

Time update: See (8),

Information update:

$$\left\{\begin{array}{l}
\text{Let } x_{k|k}^0 = x_{k|k-1}, \text{ and } \Gamma_{k|k}^0 = \Gamma_{k|k-1} \\
\text{For } i = 1 \text{ to } S \text{ do} \\
\quad \text{Sensor scheduler } i \text{ computes } \gamma_k^i \text{ based on (10)} \\
\quad \text{If } \gamma_k^i = 1 \\
\quad\quad \text{Compute } x_{k|k}^i \text{ and } \Gamma_{k|k}^i \text{ according to (9)} \\
\quad \text{Else} \\
\quad\quad \text{Generate } \tilde{y}_k^i \text{ according to Algorithm 1} \\
\quad\quad x_{k|k}^i = x_{k|k}^{i-1} + \tilde{K}_k^i \left(\tilde{y}_k^i - H^i x_{k|k}^{i-1}\right) \\
\quad\quad \Gamma_{k|k}^i = \Gamma_{k|k}^{i-1} - K_k^i H^i \Gamma_{k|k}^{i-1} \\
\quad \text{where } K_k^i = \chi\left(\zeta^i\right) \Gamma_{k|k}^{i-1} {H^i}^\top (R^i + H^i \Gamma_{k|k}^{i-1} {H^i}^\top)^{-1} \\
\quad \text{and } \tilde{K}_k^i = K_k^i, \\
x_{k|k} = x_{k|k}^S,
\end{array}\right. \tag{17}$$

where $\chi(\cdot)$ is given in (16), and $\tilde{K}_k^i$ is the gain associated with synthetic measurement $\tilde{y}_k^i$.

In the KF-SM, the choice $\tilde{K}_k^i = K_k^i$ is made to stabilize the filter, as shown later in Section IV.

Remark 1. *Note that the error covariance update of the KF-SM in* (17) *neglects an additional term caused by the randomness of the synthetic measurement. As a consequence, the KF-SM is a MMSE estimator conditioned on the synthetic measurement (as stated in Proposition 1). We show that the estimation error dynamics of the KF-SM is still input-to-state stable (i.e., the filter will not be destabilized by dropping the extra term and being over-confident on the estimation accuracy). In addition, we illustrate in the numerical experiments that the error covariance inflation caused by the synthetic measurement is in fact small.*

IV. PERFORMANCE ANALYSIS OF THE KF-SM

A. Approximate MMSE Estimator

Based on the error covariance update equation in (17), a necessary condition to minimize the trace of the error covariance is $\tilde{K}_k^i = \mathbf{0}$. However, the synthetic measurement does not help update the state estimate if $\tilde{K}_k^i = \mathbf{0}$. Hence, as shown in the next theorem, the error covariance with the minimum trace is derived under the condition that $\tilde{K}_k^i$ is assumed to be given first (i.e., the gain $\tilde{K}_k^i$ is not treated as a variable to be optimized to obtain the MMSE estimator).

Proposition 1 (Approximate MMSE estimator)**.** *Under Gaussian assumption* (13)*. The KF-SM is an approximate MMSE estimator conditioned on any pair of $\tilde{y}_k^i$ and $\tilde{K}_k^i$.*

Proof. When $\gamma_k^i = 1$, the information update of the KF-SM is the same as the standard KF, which is a MMSE estimator. Hence, we prove for the case when $\gamma_k^i = 0$.

The update of the state estimate given the synthetic measurement $\tilde{y}_k^i$ can be written as

$$\begin{aligned} x_{k|k}^i &= x_{k|k}^{i-1} + \tilde{K}_k^i \left(\tilde{y}_k^i - H_k^i x_{k|k}^{i-1}\right) \\ &= x_{k|k}^{i-1} + \tilde{K}_k^i \left(G_k^i\right)^{-1} \tilde{\epsilon}_k^i. \end{aligned}$$

Also note that conditioned on $\mathcal{I}_k^{i-1}, \gamma_k^i = 0$, $\tilde{\epsilon}_k^i$ and $\tilde{K}_k^i$,

$$\begin{aligned} & \mathbb{E}\left[x_k + \tilde{K}_k^i \left(G_k^i\right)^{-1} \tilde{\epsilon}_k^i \left| \mathcal{I}_k^{i-1}, \gamma_k^i = 0, \tilde{\epsilon}_k^i, \tilde{K}_k^i \right.\right] \\ = \; & \mathbb{E}\left[x_k \left| \mathcal{I}_k^{i-1}, \gamma_k^i = 0, \tilde{\epsilon}_k^i, \tilde{K}_k^i \right.\right] + \tilde{K}_k^i \left(G_k^i\right)^{-1} \tilde{\epsilon}_k^i \\ = \; & \mathbb{E}\left[x_k \left| \mathcal{I}_k^{i-1}, \gamma_k^i = 0 \right.\right] + \tilde{K}_k^i \left(G_k^i\right)^{-1} \tilde{\epsilon}_k^i \\ = \; & x_{k|k}^{i-1} + \tilde{K}_k^i \left(G_k^i\right)^{-1} \tilde{\epsilon}_k^i, \end{aligned} \tag{18}$$

where the second equation is derived since $\tilde{\epsilon}_k^i$ is drawn independently of x_k and x_k does not depend on $\tilde{K}_k^i$, and the third equation is due to the fact that $x_{k|k}^i = x_{k|k}^{i-1}$ without $\tilde{\epsilon}_k^i$ (i.e., without the application of synthetic measurements) when $\gamma_k^i = 0$. Hence, the trace of the estimation error covariance of the KF-SM reads

$$\begin{aligned} & \operatorname{tr}\left(\Gamma_{k|k}^i\right) \\ = & \operatorname{tr}\left(\operatorname{Cov}\left(x_k \left| \mathcal{I}_k^{i-1}, \gamma_k^i = 0, \tilde{y}_k^i, \tilde{K}_k^i \right.\right)\right) \\ = & \operatorname{tr}\left(\operatorname{Cov}\left(x_k + \tilde{K}_k^i \left(G_k^i\right)^{-1} \tilde{\epsilon}_k^i \left| \mathcal{I}_k^{i-1}, \gamma_k^i = 0, \tilde{\epsilon}_k^i, \tilde{K}_k^i \right.\right)\right) \\ = & \operatorname{tr}\left(\mathbb{E}\left[\left(x_{k|k}^{i-1} - x_k\right)\left(x_{k|k}^{i-1} - x_k\right)^\top \left| \mathcal{I}_k^{i-1}, \gamma_k^i = 0 \right.\right]\right), \end{aligned}$$

where the second equation holds since G_k^i is deterministic given $\mathcal{I}_k^{i-1}$, and the third equation is derived from (18) and the fact that x_k does not depend on $\tilde{y}_k^i$ or $\tilde{K}_k^i$. Notice that the last equation coincides with the trace of the estimation error covariance of the KF-DT in (15). Since the KF-DT is an approximate MMSE estimator under Gaussian assumption (13), the KF-SM is also an approximate MMSE estimator, for any pair of $\tilde{y}_k^i$ and $\tilde{K}_k^i$. $\square$

Since the information update of the estimation error covariance in the KF-SM is the same as the KF-DT, the stability result (Proposition 2 in [19]) also applies to the KF-SM (i.e., a critical threshold exists, such that the estimation error covariance is bounded if the applied threshold is less than the critical threshold). While [19] considers a linear time-invariant system, it can be extended to the time-varying system given the following assumptions listed in Section I-A: (*i*) The total number of possible matrices A_k and Q_k across all k are finite; and (*ii*) the total number of sensors S is finite. Under the above assumptions, the technique in [24] to show the stability of the KF for an arbitrary switching sequence can be applied to show the boundedness of the error covariance. Consequently, the error covariance of the KF-SM in a time-varying system is also bounded if the thresholds applied in the sensor schedulers are below the critical threshold. Also note that all the results shown later in this work use the fact that the error covariance is bounded.

B. Input-to-state Stability of the KF-SM

This subsection shows that the estimation error dynamics of the KF-SM is input-to-state stable, if we treat the error from the synthetic measurements (with respect to the true measurements) as an input to the system. Because the KF-SM is *input-to-state stable* (ISS), the estimation error of the KF-SM is guaranteed to be bounded if the disparity between the synthetic and the true measurements is bounded.

When $\gamma_k^i = 0$, the estimate $x_{k|k}^i$ and the estimation error $\eta_{k|k}^i$ after processing the i^{th} information update step reads

$$\begin{aligned} x_{k|k}^i &= x_{k|k}^{i-1} + \tilde{K}_k^i\left(y_k^i - H^i x_{k|k}^{i-1}\right) + \tilde{K}_k^i\left(\tilde{y}_k^i - y_k^i\right), \\ \eta_{k|k}^i &= \left(I - \tilde{K}_k^i H^i\right)\eta_{k|k}^{i-1} + \tilde{K}_k^i u_k^i + \tilde{K}_k^i v_k^i. \end{aligned}$$

Combining the above equation with time update (8) and the S recursion steps stated in (17), the error dynamics of $\eta_{k|k}$ is written as (recall that in the KF-SM the synthetic gain is chosen as $\tilde{K}_k^i = K_k^i$ when $\gamma_k^i = 0$)

$$\begin{aligned} \eta_{k|k} = &\textstyle\prod_{i=0}^{S-1}\left(I - K_k^{S-i}H^{S-i}\right)A_{k-1}\eta_{k-1|k-1} \\ &+ \textstyle\sum_{i\in\mathcal{J}_k^c}\prod_{l=0}^{S-i-1}\left(I - K_k^{S-l}H^{S-l}\right)K_k^i u_k^i \\ &+ \textstyle\sum_{i=1}^{S}\prod_{l=0}^{S-i-1}\left(I - K_k^{S-l}H^{S-l}\right)K_k^i v_k^i \\ &- w_{k-1} \end{aligned} \tag{19}$$

where $\mathcal{J}_k^c$ is the complement of $\mathcal{J}_k$ (i.e., $\mathcal{J}_k^c$ is the set of sensors that do not send measurements at time k). Note that the perturbations from the model and measurement noise (i.e., the last two terms in the right hand side of (19)) are zero-mean, and will not destabilize the error dynamics (19). Hence for the remainder of this work, we study the following unperturbed error dynamics given by the KF-SM:

$$\begin{aligned} \eta_{k|k} = &\textstyle\prod_{i=0}^{S-1}\left(I - K_k^{S-i}H^{S-i}\right)A_{k-1}\eta_{k-1|k-1} \\ &+ \textstyle\sum_{i\in\mathcal{J}_k^c}\prod_{l=0}^{S-i-1}\left(I - K_k^{S-l}H^{S-l}\right)K_k^i u_k^i, \end{aligned} \tag{20}$$

where the second term in the right hand side is not zero-mean and can potentially destabilize the system. Consequently, it is considered as an input to the error dynamics (20) rather than perturbation. Also note that (20) is treated as the evolution equation of $\eta_{k|k}$ conditioned on $\mathcal{J}_k^c$. Hence, if (20) is shown to be ISS under all $\mathcal{J}_k^c$, the ISS of the estimation error dynamics given by the KF-SM is obtained.

For all $\mathcal{J}_k^c$, the unforced system of (20) (i.e., system (20) after setting $u_k^i = 0$ for all i and k) reads

$$\eta_{k|k} = \prod_{i=1}^{S}\left(I - K_k^i H_k^i\right)A_{k-1}\eta_{k-1|k-1}. \tag{21}$$

We need to show that the unforced system (21) is *globally exponentially stable* (GES). Due to Lemma 4.6 in [25], the global exponential stability of the unforced system (21) is a sufficient condition of the ISS of the system (20).

To show the GES of (21), it suffices to prove that a Lyapunov function of the estimation error given in (21) exists such that its one-step change is negative definite. However, the conventional Lyapunov function candidate $V(k,\eta_{k|k}) = \eta_{k|k}^\top\Gamma_{k|k}^{-1}\eta_{k|k}$ is not applicable since the information update of $\Gamma_{k|k}$ in the KF-SM is different from the standard KF. Hence, we construct a virtual filtering process, namely an *auxiliary KF* (KF-AUX), which has the same estimation error and as the unforced system (21) for all k. Hence, showing the error dynamics (21) is GES is equivalent to showing the GES of the error dynamics given by the KF-AUX. It is shown in Proposition 2 that constructing a Lyapunov function for the error dynamics of the KF-AUX is straightforward.

Definition 3. The KF-AUX associated with the KF-SM is constructed as

$$\text{Time update:}\begin{cases} \breve{x}_{k|k-1} = A_{k-1}\breve{x}_{k-1|k-1} \\ \breve{\Gamma}_{k|k-1} = A_{k-1}\breve{\Gamma}_{k-1|k-1}A_{k-1}^\top + Q_{k-1}, \end{cases}$$

Information update:

$$\begin{cases} \text{Let } \breve{x}_{k|k}^0 = \breve{x}_{k|k-1}, \text{ and } \breve{\Gamma}_{k|k}^0 = \breve{\Gamma}_{k|k-1} \\ \text{For } i = 1 \text{ to } S \text{ do} \\ \quad \text{Let } \gamma_k^i \text{ be the transmission decision of the KF-SM} \\ \qquad \text{associated with sensor } i \text{ at time } k \\ \quad \breve{x}_{k|k}^i = \breve{x}_{k|k}^{i-1} + \breve{K}_k^i\left(y_k^i - H_k^i\breve{x}_{k|k}^{i-1}\right) \\ \quad \breve{\Gamma}_{k|k}^i = \breve{\Gamma}_{k|k}^{i-1} - \breve{K}_k^i H_k^i\breve{\Gamma}_{k|k}^{i-1} \\ \quad \text{Where if } \gamma_k^i\text{=}1 \\ \qquad \breve{K}_k^i = \gamma_k^i\breve{\Gamma}_{k|k}^{i-1}H_k^{i\top}\left(R_k^i + H_k^i\breve{\Gamma}_{k|k}^{i-1}H_k^{i\top}\right)^{-1} \\ \quad \text{Else} \\ \qquad \breve{K}_k^i = \breve{\Gamma}_{k|k}^{i-1}H_k^{i\top}\left(\breve{R}_k^i + H_k^i\breve{\Gamma}_{k|k}^{i-1}H_k^{i\top}\right)^{-1} \\ \breve{x}_{k|k} = \breve{x}_{k|k}^S, \end{cases}$$

where $\breve{R}_k^i = \frac{1}{\chi(\zeta_k^i)}(R_k^i + H_k^i\breve{\Gamma}_{k|k}^{i-1}H_k^{i\top}) - H_k^i\breve{\Gamma}_{k|k}^{i-1}H_k^{i\top}$. The initial guesses are set as $\breve{\eta}_{0|-1} = \eta_{0|-1}$ and $\breve{\Gamma}_{0|-1} = \Gamma_{0|-1}$.

Proposition 2 (ISS of the KF-SM)**.** *The estimation error dynamics of the KF-SM is input-to-state stable.*

Proof. As discussed earlier, it suffices to prove that system (21) is GES. One may note that by construction, in the unforced system (21) and the KF-AUX, $K_k^i = \breve{K}_k^i$ for all $i \in \mathcal{S} \cup \{0\}$ and k, hence $\Gamma_{k|k}^i = \breve{\Gamma}_{k|k}^i$ and $x_{k|k}^i = \breve{x}_{k|k}^i$ for all $i \in \mathcal{S} \cup \{0\}$ and k. Therefore, showing the GES of (21) is equivalent to showing the GES of the error dynamics given by the KF-AUX:

$$\begin{aligned}\breve{\eta}_{k|k} &= \textstyle\prod_{i=1}^{S} \left(I - \breve{K}_k^i H_k^i\right) A_{k-1} \breve{\eta}_{k-1|k-1} \\ &= \left(I - \breve{K}_k H\right) A_{k-1} \breve{\eta}_{k-1|k-1},\end{aligned}$$

where $\breve{K}_k = \breve{\Gamma}_{k|k-1} H^\top (\breve{R}_k + H \breve{\Gamma}_{k|k-1} H^\top)^{-1}$ is the Kalman gain with simultaneous processing of the sensor data, and $\breve{R}_k = \mathrm{diag}_{i \in \mathcal{S}} \left(\gamma_k^i R_k^i + (1-\gamma_k^i) \breve{R}_k^i\right)$.

Consider the following Lyapunov function candidate

$$\breve{V}(k, \breve{\eta}_{k|k}) = \breve{\eta}_{k|k}^\top \breve{\Gamma}_{k|k}^{-1} \breve{\eta}_{k|k}, \tag{22}$$

and let $\breve{F}_k = I - \breve{K}_k H_k$, the one step-change of the Lyapunov function candidate (22) is given by

$$\begin{aligned}& \breve{V}(k+1, \breve{\eta}_{k+1|k+1}) - \breve{V}(k, \breve{\eta}_{k|k}) \\ = \; & -\breve{\eta}_{k|k}^\top \left(\breve{\Gamma}_{k|k}^{-1} - A_k^\top \breve{F}_{k+1}^\top \breve{\Gamma}_{k+1|k+1}^{-1} \breve{F}_{k+1} A_k\right) \breve{\eta}_{k|k}, \\ = \; & -\breve{\eta}_{k|k}^\top \left(\breve{\Gamma}_{k|k}^{-1} - A_k^\top (A_k \breve{\Gamma}_{k|k} A_k^\top + \breve{W}_k)^{-1} A_k\right) \breve{\eta}_{k|k},\end{aligned}$$

where $\breve{W}_k = Q_k + \breve{\Gamma}_{k+1|k} H^\top \breve{R}_k^{-1} H \breve{\Gamma}_{k+1|k}$, and the last equation is due to Lemma 2 in [26]. Following the arguments of Lemma 3 in [26] where the matrix inversion lemma is applied to show the negative definiteness of one-step change of the Lyapunov function candidate, we obtain

$$\breve{\Gamma}_{k|k}^{-1} - A_k^\top (A_k \breve{\Gamma}_{k|k} A_k^\top + \breve{W}_k)^{-1} A_k > 0.$$

Therefore the error dynamics of the KF-AUX is GES, which means that equivalently the original unforced system (21) is GES. Hence, the system (20) is input-to-state stable for all $\mathcal{J}_k^c$, thus the estimation error dynamics of the KF-SM is input-to-state stable. □

Corollary 1 (Ultimate boundedness of the KF-SM)**.** *The estimation error of the KF-SM in the unperturbed error dynamics* (20) *is ultimately bounded.*

Proof. Since the error covariance $\Gamma_{k|k}$ is bounded for all k, it follows that K_k^i and G_k^i are bounded for all i and k. Also recall from Section III that $\left\|u_k^i\right\| < 2\zeta^i \left\|G_k^i\right\|^{-1} (m^i)^{\frac{1}{2}}$, where m^i is the dimension of sensor data y_k^i. Hence, there exists $L > 0$ such that the second term in (20) satisfies

$$\left\| \sum_{i \in \mathcal{J}_k^c} \prod_{l=0}^{S-i-1} \left(I - K_k^{S-l} H^{S-l}\right) K_k^i u_k^i \right\| < L \rho_0\left(\zeta_{\max}\right), \tag{23}$$

where $\zeta_{\max} = \max_{i \in \mathcal{S}} \zeta^i$, and $\rho_0(\cdot)$ is a class $\mathcal{K}_\infty$ function. Given the exponential stability of the unforced system (21),

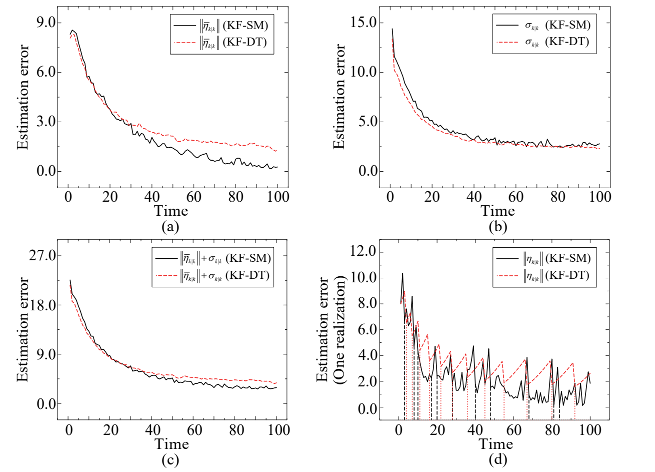

Fig. 2. (a) The 2-norm of the average estimation error $\|\bar{\eta}_{k|k}\|$ given by the KF-SM and KF-DT; (b) The variance of the estimation error $\sigma_{k|k}$ given by the KF-SM and KF-DT; (c) The sum of the 2-norm and the variance of the estimation error $\|\bar{\eta}_{k|k}\| + \sigma_{k|k}$ given by the KF-SM and KF-DT; (d) The 2-norm of the estimation error $\|\eta_{k|k}\|$ given by the KF-SM and KF-DT for a single run, where the vertical lines indicate the time steps when the sensor data is sent (with dashed and dotted vertical lines representing the KF-SM and KF-DT, respectively).

the estimation error (20) of the KF-SM satisfies

$$\|\eta_{k|k}\| \leq a b^k \|\eta_{0|0}\| + \rho\left(\sup_{0 \leq \kappa \leq k} \left\| \textstyle\sum_{i \in \mathcal{J}_\kappa^c} \prod_{l=0}^{S-i-1} \left(I - K_\kappa^{S-l} H^{S-l}\right) K_\kappa^i u_\kappa^i \right\|\right),$$

where $a > 0$, $0 < b < 1$, and $\rho(\cdot)$ is a class $\mathcal{K}$ function. Substituting (23) in the above equation, we obtain

$$\|\eta_{k|k}\| \leq a b^k \|\eta_{0|0}\| + \rho\left(L \rho_0\left(\zeta_{\max}\right)\right).$$

Hence for all $c > \rho\left(L \rho_0\left(\zeta_{\max}\right)\right)$, there exists $T(\|\eta_{0|0}\|, c)$ such that $\|\eta_{k|k}\| < c$ when $k > T(\|\eta_{0|0}\|, c)$, which yields the ultimate boundeness of the estimation error. □

V. Numerical Experiments

This section illustrates the capability of the synthetic measurements to improve the overall estimation accuracy compared to the KF-DT, and shows that the error covariance inflation caused by the synthetic measurements can be small.

The target to be estimated is a two-dimensional system, where

$$A = \begin{pmatrix} 1.05 & 0 \\ 0.1 & 0.9 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 \end{pmatrix},$$

with $Q = \mathrm{diag}\,(0.01, 1)$, and $R = 1$. In this experiment, the KF-SM and KF-DT are run for 100 times. For each run, the sensor-to-estimator communication rate is defined in (7). We investigate the estimation error provided by the KF-SM and KF-DT for a low communication rate $\bar{r} = 0.125$, where $\bar{r}$ is the average of r among 100 runs. To meet the required communication rate, the (time-invariant) deterministic threshold ζ is chosen to be 0.7 and 0.6 for the KF-SM and KF-DT, respectively. The initial state $x_0 = (2 \quad 1)^\top$, and the initial estimate is set to be $x_{0|-1} = (10 \quad 1.05)^\top + w_{0|-1}$ with $w_{0|-1} \sim \mathcal{N}\left(\mathbf{0}, 100 I_2\right)$.

Figure 2a-c plot the evolution of $\|\bar{\eta}_{k|k}\|$, $\sigma_{k|k}$, and $\|\bar{\eta}_{k|k}\| + \sigma_{k|k}$ given by the KF-SM and KF-DT, where $\bar{\eta}_{k|k} =$

$\frac{1}{100}\sum_{\tau=1}^{100}\eta_{k|k}^{\tau}$, with $\eta_{k|k}^{\tau}$ the estimation error given by the τ^{th} run at time k, and $\sigma_{k|k} = \sqrt{\frac{1}{100}\sum_{\tau=1}^{100}\|\eta_{k|k}^{\tau} - \bar{\eta}_{k|k}\|^2}$ quantifies the spread of the estimation error among different runs. After time step 50, the error covariance for the two filters converges to their fixed values, hence in the deterministic sensor scheduler, the innovation will be normalized by a fixed value (see (11)), and compared with a threshold which is fixed as well. Therefore, data transmission for the KF-SM and KF-DT will be stopped by the sensor schedulers when the estimation error reaches a fixed value (i.e., around $\| \eta_{k|k} \| = 2$ for the KF-DT as shown in Figure 2d). For the KF-DT, it is impossible for the estimation error to further decrease when the sensor stops sending data. Comparatively, the estimation error of the KF-SM can continue to decrease, because the estimation error can still decrease even when the sensor data is not sent due to the feedback provided by the synthetic measurements (as shown in Figure 2d). Consequently, the KF-SM in Figure 2a has a smaller average estimation error $\|\bar{\eta}_{k|k}\|$ compared to the KF-DT.

Figure 2b shows that the variances $\sigma_{k|k}$ of the estimation error of the KF-SM is slightly larger than the KF-DT. The randomness of the estimation error for the KF-DT is given by the modeling error and measurement noise, while for the KF-SM it is given by the modeling error, measurement noise, and the synthetic measurement. Hence, the KF-SM trades the variance $\sigma_{k|k}$ in favor of reducing the average estimation error $\bar{\eta}_{k|k}$. Nevertheless, the reduction on $\|\bar{\eta}_{k|k}\|$ for the KF-SM can dominate the increase of $\sigma_{k|k}$, which is shown in Figure 2c.

VI. CONCLUSIONS

In this article, we propose an information update strategy for the KF under a deterministic threshold-based sensor scheduler, where synthetic measurements are generated based on the sensor scheduler and fed back into the state estimate to promote the reduction of the estimation error when data transmission is not triggered. We prove that under the Gaussian assumption of the state estimate, the proposed KF-SM is an approximate MMSE estimator when the synthetic measurement is given. We also show the error dynamics of the estimate is input-to-state stable, and the estimation error of the KF-SM is ultimately bounded. Numerical experiments show an advantage of the KF-SM on reducing the estimation error compared to the case without synthetic measurements. The theoretical analysis of the advantage of synthetic measurements on improving the estimation quality, as well as the error covariance inflation caused by the randomness of the synthetic measurements will be studied in our future work.

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