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Stabilizing Traffic Flow via a Single Autonomous Vehicle: Possibilities and Limitations

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Abstract— **In certain flow regimes, the ideal uniform vehicle flow on the road is unstable, and stop-and-go traffic develops. The instability that leads to this less fuel-efficient unsteady flow results from the collective behavior of all human drivers. This work studies under which circumstances the presence of a single *autonomous vehicle* (AV) can locally stabilize the flow, without changing the way the humans drive. If possible, this can enable traffic flow control via very few AVs serving as mobile actuators. First, the analysis of car-following models reveals that in idealized conditions (no system noise), the flow can in fact be made linearly stable by means of a low fraction of control vehicles. Second, we highlight the fundamental limitations of this sparse control when considering models with noise.**

I. Introduction

Despite substantial developments in road design and traffic control systems, the basic human driving behavior has not changed fundamentally over the last 80 years. However, soon the behavior of the individual agents on the roadways is about to change due to the introduction of increasingly connected and level 1 and above *autonomous vehicles* (AVs). While AVs should drive similar to humans, their actions should differ from those of humans in that they are supposed to be safer and more efficient. Once all vehicles of the roadways are autonomous, the task of traffic flow control becomes a robotics design problem, with human agents removed. However, before that, a long period of mixed human–AV flow will occur on our roadways. Numerous simulation-based studies have demonstrated that at AV penetration rates above 20%, the insertion of AVs can have substantial benefits on the stability properties of traffic flow, see Section I-A. However, before such penetration rates are achieved, a small number of AVs will travel on our roadways. This paper focuses on the possibilities, as well as the limitations, that arise with having few ($< 5\%$) AVs on the road.

It can be observed [1], and has been demonstrated experimentally [2], [3], [4], that in certain flow regimes uniform flow can turn unsteady, dominated by traffic waves. This "phantom traffic jam" phenomenon can be explained via instabilities in traffic models: macroscopic [5], cellular [6], and microscopic [7]. The instability, and the characteristics of the emerging nonlinear waves, are caused by the collective behavior of the human agents on the road. It can also be shown that effects like lane changing, road features, ramps, or vehicle/driver inhomogeneities can amplify these phenomena; however, the basic occurrence of traffic waves "out of the nothing" can arise even if all agents on the road are identical, behave fully predictably, and by the same principles.

To establish a theoretical framework for the analysis of control of traffic flow via a single AV, we consider a ring road setup, i.e., a circular single-lane roadway, on which n vehicles are placed, one of which is an AV. This setup can be interpreted as a model for an infinite road, on which every n-th vehicle is an AV, i.e., the AV penetration rate is $1/n$. The experimental validation of traffic waves growing out of an unstable uniform flow was conducted on a ring road [2].

A. Prior research on traffic control via AVs

The theoretical foundations of traffic flow stability go back to the first car-following models [8], [9]. An overview is provied by Wilson and Ward [10], outlining string stability [11] and related stability concepts. For a car-following model on a ring road, linearized around a uniform flow equilibrium state, most stability criteria, including string stability, are equivalent to the system possessing no growing eigenmodes.

Research on the control of vehicular traffic via autonomous vehicles has focused mostly on platoon control or mixed human–AV traffic. In platooned traffic, all vehicles on the roadway are considered equipped with the same control and communication technology, and can be controlled centrally to achieve stability. This ideal situation allows for the AVs to travel with extremely short spacings, while maintaining stable flow dynamics. Stability results when all AVs use the same *adaptive cruise controller* (ACC) or *connected adaptive cruise controller* (CACC) are given in [12], [13], [14] and [15], [16], [17], respectively. While AV-only systems are of great interest for the far future, the near future will confront us with mixed human–automated traffic flow (AVs or ACC-enabled vehicles). This situation is more complicated in terms of a theoretical stability analysis, and most research on the system stability has been simulation-based [18], [19], [20], [16].

Important results on fully ACC-equipped platoon traffic are the works by Ioannou et al. [13], whose throttle and break controller (based on a constant time headway) achieves string stability, which is demonstrated both in simulation and in a field experiment. Similarly, Liang and Peng [18] guarantee string stability via a controller that uses headway and its rate of change as input to the control. The performance of this controller is then investigated under different ACC pen-

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etration rates. Darbha and Rajagopal [14] demonstrate that constant headway policies may lead to unstable traffic flows in platooned systems, and suggest an alternative controller framework that is devoid of those instabilities.

Important simulation-based studies include the work by Kesting et al. [20], who investigate the throughput of roads at various levels of ACC penetration rates, using the *intelligent driver model* (IDM) [21]. The IDM is used for both human-controlled and ACC vehicles, but with different parameters. The (in)stability of traffic flow is not explicitly considered. Telabpour and Mahmassani [16] use simulations to study the mixed human–AV flow and particularly investigate the stability of such flow. The study is limited to the medium to high AV penetration rate regime. In another study, Rajamani and Zhu [15] introduce *semi–autonomous adaptive cruise control* (SAACC), which is based on a headway that strongly changes with velocity. One drawback of such controls is that on real highways, large gaps would encourage undesirable cut-ins, i.e., other vehicles may change lanes to fill the gap ahead of the control vehicle.

Formal stability results on mixed human-AV traffic flow have recently been provided by Talebpour [22], based on a stability analysis framework by Ward [23]. Like in [20], both human-driven and AVs are assumed to follow the IDM, but with different parameters. Jin and Orosz [24] show that a vehicle can use a CACC controller with information from human-piloted vehicles upstream to achieve strong-stable traffic conditions.

B. Contributions of this research

The extreme case of a single AV, surrounded by many human-controlled vehicles, has not been the focus of these prior studies. Moreover, analytical stability results for mixed flow have been established only for systems without noise, while investigations with system noise have been simulation-based. Here we investigate the practically challenging single-AV case, and provide analytical stability results in the presence of system noise. In Section II, linear stability results of car-following models are presented. Those results are then extended in Section III to the situation of a single AV present, and controllability results are established in Section IV. In Section V we argue that the theoretical stabilizability results cannot properly describe reality, and we resolve the apparent paradox. The paper closes with a discussion in Section VI.

II. Stability of Traffic Without Control

We consider a circular road (single lane, no ramps, uniform road conditions) of length L, and place n identical vehicles on it. The reasons for considering a ring road are: (*i*) no boundary conditions are needed, and one has a perfect control on the average vehicle density; (*ii*) the circular road corresponds to an infinite straight road with n-periodic traffic dynamics; (*iii*) experimental evidence of instabilities exists [2], and the experimental results can be used to calibrate model parameters.

Let x denote the position along the road (defined modulo L), and t time. The position of the j-th vehicle is $x_j = x_j(t)$, and its velocity and acceleration are $\dot{x}_j = \frac{\mathrm{d}}{\mathrm{d}t} x_j$ and $\ddot{x}_j = \frac{\mathrm{d}^2}{\mathrm{d}t^2} x_j$, respectively. Let the vehicles be ordered so that vehicle $j+1$ precedes (leads) vehicle j (with n-periodic wrap-around, i.e., vehicle 1 precedes vehicle n. We consider second-order car-following models, in which the acceleration of each vehicle,

$$\begin{aligned} \ddot{x}_j &= f(x_{j+1} - x_j, \dot{x}_{j+1} - \dot{x}_j, \dot{x}_j) \\ &= f(h_j, \dot{h}_j, v_j) \ , \end{aligned} \tag{1}$$

depends only on its spacing $h_j = x_{j+1} - x_j$, its relative speed to the vehicle ahead, $\dot{h}_j = \dot{x}_{j+1} - \dot{x}_j$, and its own velocity $v_j = \dot{x}_j$. Note that the distance h_j between vehicles is measured from vehicle front to front. Hence, for realistic traffic, $h_j = \ell$ would correspond to vehicle collisions, where ℓ is the vehicle length, and traffic models should ideally be designed to prevent collisions. This study focuses on the near-equilibrium flow, which is far from collisions. The acceleration function f defines the model dynamics. Note that, even though each vehicle only uses information from the vehicle immediately ahead, due to the periodicity of the road, the dynamics of all vehicles are fully inter-dependent.

We restrict to such functions f that have the property that the model (1) admits exactly one equilibrium state (given L and n), in which all vehicles move with the same speed v_{eq} and are equi-spaced (at distance $h_{\text{eq}} = L/n$). At equilibrium, vehicle accelerations are zero, as are changes in relative velocity. Therefore, the equilibrium velocity v_{eq} is so that $f(h_{\text{eq}}, 0, v_{\text{eq}}) = 0$.

One important example of a car-following model that satisfies the assumptions is the *optimal-velocity-follow-the-leader* (OV-FTL) model

$$\ddot{x}_j = a \cdot (V(h_j) - \dot{x}_j) + b \cdot \dot{h}_j/(h_j)^2 \ , \tag{2}$$

in which a vehicle's acceleration is affected by (*i*) how much its own velocity differs from an optimal velocity $V(h_j)$, and (*ii*) it relative velocity to the vehicle ahead. The optimal velocity function $V(\cdot)$ is an increasing function of the spacing h, and the effect of the relative velocity decreases with the spacing. This last aspect prevents trajectories from crossing, even in the presence of instabilities.

We now consider system (1) under small perturbations from the equilibrium flow, i.e., we define $x_j = x_{eq} + y_j$ and $v_j = v_{eq} + u_j$, where y_j and u_j are the infinitesimal position and velocity difference deviations, respectively. Substituting these expressions into the model (1) yields the linearized system

$$\ddot{y}_j = \alpha_1 \left(y_{j+1} - y_j\right) - \alpha_2 \, u_j + \alpha_3 \, u_{j+1} \ , \tag{3}$$

which is defined by the three parameters $\alpha_1 = \frac{\partial f}{\partial h}$, $\alpha_2 = \frac{\partial f}{\partial \dot{h}} - \frac{\partial f}{\partial v}$, and $\alpha_3 = \frac{\partial f}{\partial \dot{h}}$. All partial derivatives are evaluated at the equilibrium state.

The linear dynamics (3) can be expressed in matrix form

$$\dot{\mathbf{z}} = \mathbf{M} \cdot \mathbf{z} \ , \tag{4}$$

where $\mathbf{z} = [y_1, y_2, \ldots, y_n, \ u_1, u_2, \ldots, u_n]^T$ and

$$\mathbf{M} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{A} & \mathbf{B} \end{bmatrix} . \tag{5}$$

Here $\mathbf{O}$ and $\mathbf{I}$ are $n \times n$ zero and identity matrices, respectively, and

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & \alpha_1 & 0 & \cdots & 0 \\ 0 & -\alpha_1 & \alpha_1 & \ddots & \vdots \\ \vdots & \ddots & -\alpha_1 & \ddots & 0 \\ 0 & & \ddots & \ddots & \alpha_1 \\ \alpha_1 & 0 & \cdots & 0 & -\alpha_1 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} -\alpha_2 & \alpha_3 & 0 & \cdots & 0 \\ 0 & -\alpha_2 & \alpha_3 & \ddots & \vdots \\ \vdots & \ddots & -\alpha_2 & \ddots & 0 \\ 0 & & \ddots & \ddots & \alpha_3 \\ \alpha_3 & 0 & \cdots & 0 & -\alpha_2 \end{bmatrix}.$$

Note that many types of nonlinear car-following models lead to (3) upon linearization around their equilibrium state; the OV-FTL model (2) is just one particular example.

In the linearized model (3), we assume that the following "common sense" assumptions hold: the acceleration of a vehicle is reduced when (a) the spacing h decreases; (b) the relative velocity $\dot{h}$ decreases; or (c) the vehicle's own velocity v increases. These risk aversion criteria lead to the sign constraints $\alpha_1 > 0$, $\alpha_2 > \alpha_3$, and $\alpha_3 > 0$.

The eigenmodes of system (4) can be obtained by substituting the ansatz $y_j(t) = c_j e^{\omega t}$, where $c_j, \omega \in \mathbb{C}$, into (3), which leads to

$$c_j \omega^2 = \alpha_1 (c_{j+1} - c_j) - c_j\, \alpha_2\, \omega + \alpha_3\, c_{j+1}\, \omega \ ,$$

or equivalently $c_j = F(\omega) c_{j+1}$, where

$$F(\omega) = \frac{\alpha_1 + \alpha_3\, \omega}{\alpha_1 + \alpha_2\, \omega + \omega^2} \tag{6}$$

is the transfer function that translates the lead vehicle's motion (c_{j+1}) to its follower's (c_j). The periodicity on the circular road then requires

$$F(\omega)^n = 1 \ . \tag{7}$$

The $2n$ roots of equation (7) yield exactly the eigenvalues of system (4). In particular, if none of the roots of (7) possess positive real part, then the system (4) is stable, i.e., infinitesimal perturbations from equilibrium do not amplify and the system remains close to equilibrium.

This stability criterion depends on n, the number of vehicles on the ring. However, in the limit $n \to \infty$ (while scaling $L \propto n$, i.e., this is the infinite road limit; the one that one actually cares about in reality), the roots lie dense in a curve $\mathcal{C} \subset \mathbb{C}$, which is independent of n. This curve is given by

$$\mathcal{C} = \{z \in \mathbb{C} \,:\, |F(z)| = 1\} \ .$$

We now formulate a stability criterion for this limit case. It guarantees stability also for any finite n:

$$\text{The limit system is stable if } \mathcal{C} \subset \mathbb{C}^- . \tag{8}$$

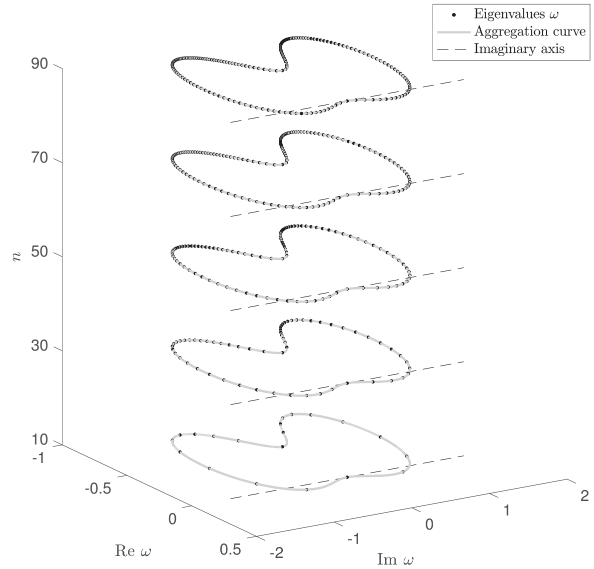

Fig. 1. Locations of system eigenvalues for various number of vehicles n (with same equilibrium state, i.e., $L \propto n$). All eigenvalues are contained on a curve $\mathcal{C}$ that is independent of n. Hence, if the curve $\mathcal{C}$ does not enter the right half plane, stability is implied for any n; with equivalence applying in the limit $n \to \infty$.

Here $\mathbb{C}^- = \{z \in \mathbb{C} \,:\, \text{Re}(z) \leq 0\}$ is the left half plane. A substantial simplification is given by the fact that (8) is equivalent to

$$|F(iv)| \leq 1 \quad \text{for} \quad v \in \mathbb{R} \ , \tag{9}$$

i.e., it suffices to study the function F along the imaginary axis. The equivalence of (8) and (9) follows from the fact that F is holomorphic in the right half plane (because all poles of $F(z)$ lie in the left half plane). Hence the maximum of $|F(z)|$ over the right half plane can only be achieved on the imaginary axis (because $|F(z)| \to 0$ as $\text{Re}(z) \to \infty$).

A straightforward evaluation of $|F(iv)|^2$ yields that (9) is equivalent to

$$\Delta_\alpha = \alpha_2^2 - \alpha_3^2 - 2\alpha_1 \geq 0 \ . \tag{10}$$

III. Stability with Control Vehicle

We now consider that one of the n vehicles on the ring road is autonomous. We do so by assuming that it follows other dynamics

$$\ddot{x}_j = g(h_j, \dot{h}_j, v_j) \ , \tag{11}$$

defined by the function g. We consider the situation that the autonomous vehicle leaves the same spacing at equilibrium as a human driver would (the rationale being that larger spacings would invite cut-in behavior on real highways; and smaller spacings would be perceived unsafe by the surrounding traffic). Moreover, we assume that the AV is able to accurately estimate the equilibrium state. And furthermore, for simplicity we assume that the AV's dynamics are simple car-following as well. Linearizing (11) around the equilibrium state yields

$$\ddot{y}_j = \beta_1\,(y_{j+1} - y_j) - \beta_2\,u_j + \beta_3\,u_{j+1}\ ,$$

where $\beta_1 = \frac{\partial g}{\partial h}$, $\beta_2 = \frac{\partial g}{\partial \dot{h}} - \frac{\partial g}{\partial v}$, and $\beta_3 = \frac{\partial g}{\partial \dot{h}}$. As in (6), we define the AV transfer function $G(\omega) = \frac{\beta_1+\beta_3\,\omega}{\beta_1+\beta_2\,\omega+\omega^2}$. When having one AV and $n-1$ human-controlled vehicles on the ring road, the periodicity implies that $F(\omega)^{n-1}G(\omega) = 1$. As in Section II, we consider the stability criterion (8), where the curve $\mathcal{C}$ is now given as

$$\mathcal{C} = \{z \in \mathbb{C}\,:\,|F(z)|^{1-\gamma}\cdot|G(z)|^{\gamma} = 1\}\ .$$

Here $\gamma = 1/n$ is the AV penetration rate, and the prior results are obtained in the case $\gamma = 0$. As above, it suffices to study the function $H_\gamma(z) = F(z)^{1-\gamma}\cdot G(z)^{\gamma}$ along the imaginary axis. Stability is obtained if $|H_\gamma(iv)| \le 1$ for $v \in \mathbb{R}$, which is equivalent to the condition

$$(1-\gamma)D_\alpha(v) + \gamma D_\beta(v) \le 0\ , \tag{12}$$

where $D_\alpha(v) = \log\left(\frac{\alpha_1^2+\alpha_3^2v^2}{(\alpha_1-v^2)^2+\alpha_2^2v^2}\right)$ and $D_\beta(v) = \log\left(\frac{\beta_1^2+\beta_3^2v^2}{(\beta_1-v^2)^2+\beta_2^2v^2}\right)$.

We now assume that the human dynamics violate the stability condition (10), i.e., $\Delta_\alpha = \alpha_2^2 - \alpha_3^2 - 2\alpha_1 < 0$, while the AV dynamics satisfy $\Delta_\beta = \beta_2^2-\beta_3^2-2\beta_1 > 0$. This implies that $D_\beta(v) < 0$ for all $v > 0$. In turn, the function $D_\alpha(v)$ switches sign: it is positive only for $0 < v < \sqrt{-\Delta_\alpha}$.

Hence, we have obtained the following result. Given the parameters α_1, α_2, α_3, β_1, β_2, β_3, condition (12) can be satisfied if γ is chosen sufficiently large. The minimum AV penetration rate γ_{min} that is needed to yield stable dynamics can in principle be determined from the functions $\Delta_\alpha(v)$ and $\Delta_\beta(v)$, albeit with complicated expressions. While this shows that AV-based stabilization of otherwise unstable traffic flow is in fact possible, it is not quite what we want here. What we need is to establish that given a (low) AV penetration rate γ, AV parameters β_1, β_2, β_3 can be found that lead to stable dynamics. This result is established next.

IV. Controllability and Stabilization

We are interested in stabilizing the traffic system by means of a single AV. Hence we write the linearized system (4) introduced in Section II as a control system

$$\begin{cases}\dot{\mathbf{z}} = \mathbf{M}\cdot\mathbf{z} + \mathbf{b}\cdot\mathbf{u},\\ \mathbf{u} = \mathbf{F}\cdot\mathbf{C}\cdot\mathbf{z}\ .\end{cases} \tag{13}$$

Here $\mathbf{b} = [0,\dots,0,1,0,\dots,0]^T$ is a vector with a single non-zero component, representing the fact that the control $\mathbf{u}$ can affect the state $\mathbf{z}$ only via the AV's velocity. The matrix $\mathbf{C}$ encodes which state information the AV controller has accessible. And $\mathbf{F}$ is the feedback control matrix.

We now show that this system is always controllable, for any number of human-controlled vehicles n. The Hautus test states that system (13) is controllable if any left eigenvector $\mathbf{e}$ of $\mathbf{M}$ satisfies $\mathbf{e}^T\,\mathbf{b} \neq \mathbf{0}$. In fact, any eigenvector of $\mathbf{M}$ (associated with eigenvalue ω) has the form $\mathbf{e} = e^{\omega t}[c_1, c_2, \dots, c_n, \omega c_1, \omega c_2, \dots, \omega c_n]$, and therefore $\mathbf{e}\cdot\mathbf{b} = \omega c_1$. For nontrivial eigenvalues ($\omega \neq 0$), ωc_1 is nonzero. For the only zero eigenvalue, the eigenvector is $[1, 1, \dots, 1, 0, 0, \dots, 0]$, which corresponds to a global shift that does not affect the equilibrium. Thus, modulo translations of the whole traffic state, the system is controllable for any $n > 0$.

A particularly simple control can be obtained when the AV knows nothing about the other vehicles (i.e. $\mathbf{C} = [\mathbf{0}, \mathbf{0}, \dots, \mathbf{0}, \mathbf{1}, \mathbf{0}, \dots, \mathbf{0}]$), except for the equilibrium flow velocity v_{eq}. It turns out that the system remains controllable for any n, and thus stabilizable, meaning that a single well-designed autonomous vehicle is able to shift the eigenvalues to the left half plane.

A simple manifestation of said control is simply to choose $\beta_1 = \alpha_1$, $\beta_3 = \alpha_3$, and to increase β_2 as much as necessary. For example, the nonlinear OV-FTL model (2) can be slightly augmented for the AV

$$\ddot{x}_j = a\cdot(V(h_j) - \dot{x}_j) + b\cdot\dot{h}_j/(h_j)^2 + c\,(v_{\text{eq}} - \dot{x}_j)\ , \tag{14}$$

where the last term is a feedback control that relaxes the autonomous vehicle to the equilibrium speed v_{eq}, and c denotes the gain of this control.

The theoretical results in Section III can be used to show that this simple control suffices to stabilize the flow dynamics for any number of vehicles n. As $n \to \infty$, the eigenvalues of the controlled system converge to two numbers r_1 and r_2 in the left half plane, given by $r_{1,2} = \frac{-\alpha_2\pm\sqrt{\alpha_2^2-4\alpha_1}}{2}$. An asymptotic calculation for $n \gg 1$ reveals that the minimum gain required to stabilize the system scales like

$$c_{\text{min}} \approx \left|\frac{r_2 - r_1}{\alpha_1 + r_1\alpha_3}\right|^n\cdot\left(\frac{2}{\alpha_2}\right)^n\cdot\frac{1}{r_2 - r_1}\ . \tag{15}$$

This means that the gain required to stabilize the dynamics grows exponentially in the number of vehicles on the road.

V. Paradox and Resolution

We have established that a single AV can stabilize otherwise unstable traffic flow, for any number of vehicles. Clearly, this result contradicts common sense. On a circular road of enormously large radius it is practically impossible to keep traffic flow, that "wants" to be unstable, at bay by means of a single control vehicle. How can this paradox be resolved?

One may suggest that the exponential growth (in n) of the control gain (15) is enough to render the control unrealistic. However, large gains alone do not imply large forcing, because the system's deviation from equilibrium could be comparatively small.

To understand the fundamental limitations of single vehicle control for large n, one first must rationalize that the stabilization, supported by eigenmode analysis, only categorizes the behavior of the whole system as $t \to \infty$. That means that any small perturbation will eventually decay to zero; however, in that process it may produce some large deviations. Due to the nature of the car-following dynamics, perturbations get passed from vehicle to vehicle via the

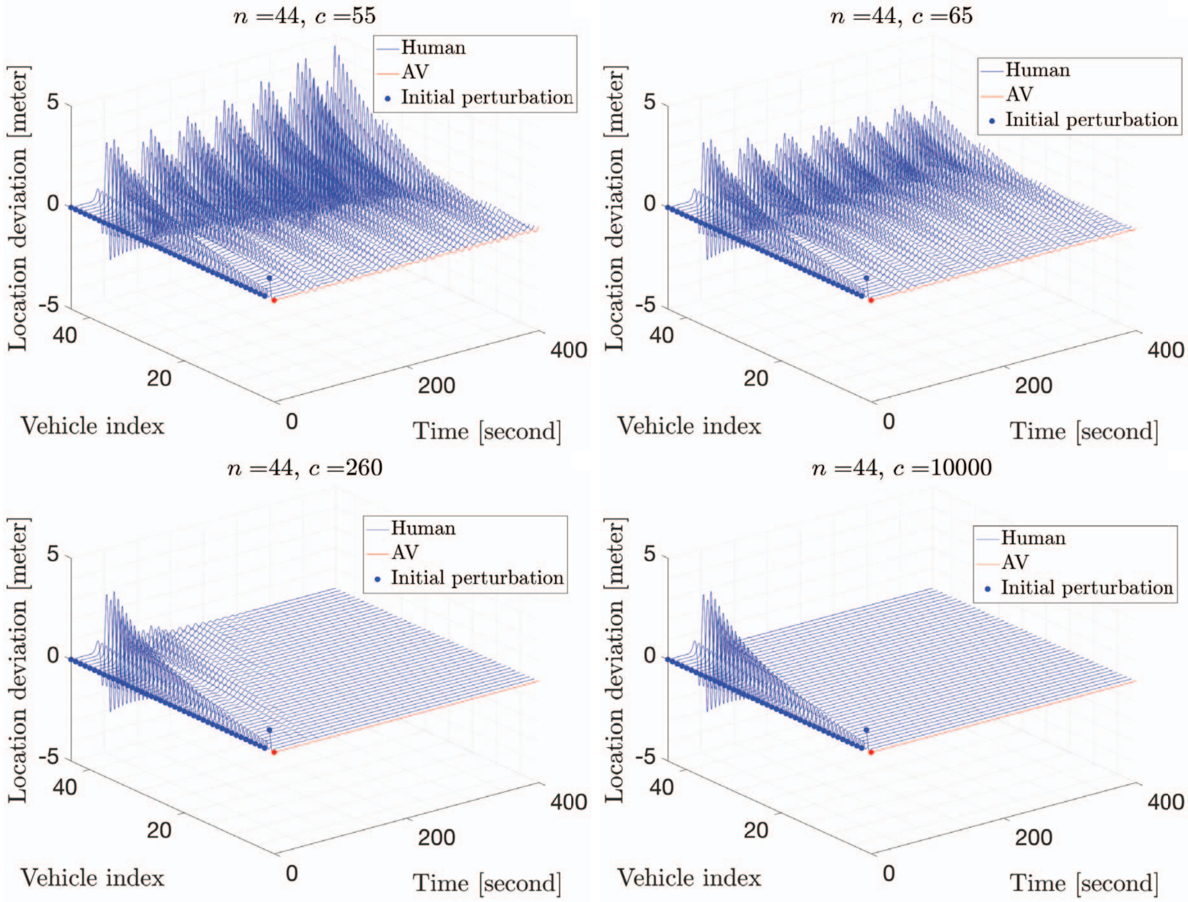

Fig. 2. Equilibrium deviations of vehicles for four manifestations of the traffic system on a circular road, representing four choices of control gains c (small to large). The AV is shown in red, and the vehicle following it is initially perturbed. The peak of perturbation, generated before the wave hits the AV, cannot be affected by the control. The parameters for the human-driver model (2) are chosen so that wave behavior resembling the experiment [2] is obtained.

transfer function (6). Instability of human-controlled traffic means that the perturbation grows in magnitude as it travels from vehicle to vehicle. And stability of the controlled system means that the AV dampens the perturbation by more than the amplification caused by the other $n-1$ vehicles. While this does ensure the eventual decay of any perturbation, perturbations may get amplified up to $n-2$ times before they are affected by the AV; and if n is large, that may result in large deviations from equilibrium. In other words, the formally stabilized system fails to remain close to equilibrium, if the model is augmented by some noise.

Figure 2 shows the results of four manifestations of the traffic system with $n = 44$, subject to the control law (14) with four different levels of gain c. At $t = 0$, a perturbation is added to the vehicle following the AV. As this perturbation cascades through the chain of human-controlled vehicles, it gets amplified; and once it hits the AV, it gets damped. From top-left to bottom-right, increasing gains have been used. While the first panel has not been stabilized (the signal grows in time), the other three panels correspond to stabilized dynamics. In particular, in the last panel the AV removes the perturbation almost completely. However, in all cases the perturbation grows to a fixed magnitude before it hits the AV, and this growth is independent of the gain. In particular, it cannot be affected via the controller.

In reality, perturbations will act upon the system at all times. Hence, signals like the one shown in Fig. 2 will be superimposed to yield the system's total deviation from equilibrium. Such persistent perturbations can be described by augmenting the model by a stochastic noise term:

$$\mathrm{d}v_j = f(h_j, \dot{h}_j, v_j)\,\mathrm{d}t + s_j\,\mathrm{d}B_t\ .$$

Here B stands for Brownian motion and s_j denotes the perturbation strength that the j-th vehicle is subjected to. After linearization, we reach

$$\mathrm{d}u_j = [\alpha_1(y_{j+1}-y_j) - \alpha_2 u_j + \alpha_3 u_{j+1}]\mathrm{d}t + s_j \mathrm{d}B_t\ . \quad (16)$$

We now need to quantify the variance of the solution of (16) to understand how far each vehicle can be expected to be from equilibrium. Using the notations in (5), the variance of the solution is

$$\mathrm{Var}\,\mathbf{z}(t) = \mathbf{R}(t)\cdot[\mathbf{0}_{1\times n}\quad s_1^2, s_s^2, \dots, s_n^2]^T\ , \quad (17)$$

where we have the perturbation strength vector $\mathbf{s}$:=

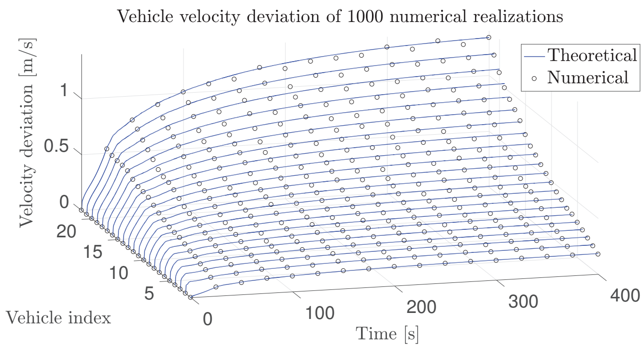

Fig. 3. Velocity standard deviations obtained by (a) approximation using 1000 numerical realizations, and (b) formula (17), up to time $t = 400$s. Every vehicle, including the AV, experiences noise of magnitude $s = 0.2$.

$[s_1, s_2, \ldots, s_n]$. The matrix $\mathbf{R}(t)$ is defined as

$$\mathbf{R}(t) = \int_0^t \left[\exp\left(\tau\mathbf{M}\right)\right]^2 \mathrm{d}\tau \ .$$

The exponential notation denotes a matrix exponential. The integration and square are conducted entry-wisely.

Figure 3 shows the velocity deviation from equilibrium in time and in distance from the AV. Monte-Carlo sampling of (16) is compared with the results obtained by formula (17). The results (*i*) confirm the theoretical expressions, and (*ii*) demonstrate that, after some initialization phase, each vehicle will establish an average deviation from equilibrium, and that deviation grows exponentially with the vehicle's distance from the AV. As a consequence, the system's deviation from equilibrium can substantially exceed the magnitude of system noise, in particular for large values of n.

VI. Conclusions

We have established theoretically that the stabilization of (human) traffic flow via a single autonomous vehicle is in principle possible. In fact, formally any number of vehicles (on a ring road) can be stabilized by means of a single AV. In practice, however, there is an upper bound of vehicles than can be stabilized. The study of a model with persistent noise reveals that, even for stabilized systems, noise gets amplified by a factor that grows exponentially in the number of vehicles. Hence, in reality (i.e., with noise) the system cannot be kept close to equilibrium, if too many human-driven vehicles are on the road. The results shown in Figures 2 and 3 indicate that the maximum number of vehicles that can be stabilized via a single AV (under realistic conditions) is likely between 20 and 40.

The findings in this article form a starting point for further development of control strategies that may influence the emergent properties of traffic long before all vehicles are connected or autonomous. In our future work, we plan to instrument and test an autonomous vehicle in a stream of human-controlled vehicles to establish an empirical limit to the stabilizing effect of a few autonomous vehicles in the traffic stream.

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